

DIFFEOMORPHISMS OF 7-MANIFOLDS WITH COCLOSED G_2 -STRUCTURE

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ABSTRACT. We introduce $\text{co}G_2$ -vector fields, coRochesterian 2-forms and coRochesterian vector fields on manifolds with a coclosed G_2 -structure as a continuation of work from [15], and we show that the spaces $\mathcal{X}_{\text{co}G_2}$ of $\text{co}G_2$ -vector fields and $\mathcal{X}_{\text{coRoc}}$ of coRochesterian vector fields are Lie subalgebras of the Lie algebra of vector fields with the standard Lie bracket. We also define a bracket operation on the space of coRochesterian 2-forms Ω_{coRoc}^2 associated to the space of coRochesterian vector fields and prove, despite the lack of a Jacobi identity, a relationship between this bracket and so-called *co* G_2 -morphisms.

INTRODUCTION

This paper is a natural continuation of the papers [15, 16] wherein we considered 7-manifolds with (closed) G_2 -structures. This is part of a program to better understand the exceptional geometries by viewing them as analogues of symplectic geometry; moreover, this philosophy can be viewed as a continuation of the works of Brown, Fernandez and Gray [22, 6, 18, 19, 21, 20] where symplectic, G_2 and $\text{Spin}(7)$ geometries all come from the study of cross products on manifolds. In particular, [20] proves that there exist 16 subclasses of G_2 -structures including closed G_2 -structures, coclosed G_2 -structures as well as G_2 -structures which are both closed and coclosed. In our previous article, we considered analogues of Hamiltonian and symplectic vector fields on manifolds with closed G_2 -structure which we called *Rochesterian* and G_2 -vector fields respectively; the current article is concerned with developing these ideas in the case of a coclosed G_2 -structure. Note that the fundamental 3-form φ is coclosed if and only if the associated 4-form $\star\varphi$ is closed. Here $\star\varphi$ is the 4-form which is Hodge-dual to φ with respect to the Hodge star operator associated to the metric defined by φ . This is in contrast to the symplectic case where the fundamental 2-form being closed is equivalent to it being coclosed.

The main purpose of this paper is to define *co* G_2 -vector fields, *coRochesterian* 2-forms and *coRochesterian* vector fields for coclosed G_2 -structures and prove the following results:

Theorem. *Every coRochesterian vector field on a manifold M with coclosed G_2 -structure φ is a $\text{co}G_2$ -vector field. If every closed form in $\Omega_7^3(M)$ is exact, then the spaces $\mathcal{X}_{\text{coRoc}}(M)$ and $\mathcal{X}_{\text{co}G_2}(M)$ coincide.*

Corollary. *If $H^3(M) = \{0\}$, then every $\text{co}G_2$ -vector field on a manifold with coclosed G_2 -structure is a coRochesterian vector field.*

We next show that the spaces of $\text{co}G_2$ - and coRochesterian vector fields admit the structure of Lie algebras with Lie bracket induced from the standard Lie bracket structure on the space of all vector fields and prove the following result on inclusions:

Proposition. *For any $\text{co}G_2$ -vector fields X_1, X_2 , $[X_1, X_2]$ is a coRochesterian vector field with associated coRochesterian 2-form given by $\star\varphi(X_2, X_1, \cdot, \cdot)$.*

Finally, we equip the space of coRochesterian 2-forms with a bracket structure analogous to that of the Poisson bracket from symplectic geometry, show that it does *not* satisfy the Jacobi identity, show that there is a linear transformation Φ of the *vector spaces* of coRochesterian 2-forms and coRochesterian vector fields and prove the following result regarding these structures:

Theorem. (1) *Given two coRochesterian 2-forms $\sigma_1, \sigma_2 \in \Omega_{\text{Roc}}^2(M)$, $\{\sigma_1, \sigma_2\} \in \ker \Phi$ if and only if $d\sigma_1$ is constant along the flow lines of X_{σ_2} if and only if $d\sigma_2$ is constant along the flow lines of X_{σ_1} .*
 (2) *Let $\psi : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ be a diffeomorphism. Then ψ is a $\text{co}G_2$ -morphism if and only if $\psi^*(\{\sigma, \tau\}) = \{\psi^*\sigma, \psi^*\tau\}$ for all $\sigma, \tau \in \Omega_{\text{coRoc}}^2(M_2)$.*

1. G_2 GEOMETRY

If we consider coordinates (x_1, \dots, x_7) on \mathbb{R}^7 , we can define a 3-form φ_0 by

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}.$$

From this 3-form, we get an induced metric and orientation by the formula

$$(X \lrcorner \varphi_0) \wedge (Y \lrcorner \varphi_0) \wedge \varphi_0 = 6 \langle X, Y \rangle_{\varphi_0} \text{dvol}_{\varphi_0}$$

for vector fields $X, Y \in \mathcal{X}(\mathbb{R}^7)$. Then, using this metric, we can define a 2-fold vector cross product of X and Y as the unique vector field $X \times Y$ satisfying $\langle X \times Y, Z \rangle_{\varphi_0} = \varphi_0(X, Y, Z)$ for all $Z \in \mathcal{X}(\mathbb{R}^7)$. This metric then gives the associated Hodge star from which we get the dual of φ_0 given by the 4-form

$$\star\varphi_0 = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.$$

Here, we have started with the 3-form and have shown how to define the other structures in terms of it; a fundamental fact of G_2 -geometry however is that given *any one* of $\varphi_0, \star\varphi_0, \times$ or \langle, \rangle , we can always define the other structures. References for this information and an equivalent formulation of these structures arising from the octonions include [6, 7, 20, 22, 24, 25, 26].

Definition 1.1. A manifold M is said to have a G_2 -structure if there is a 3-form φ such that $(T_p M, \varphi) \cong (\mathbb{R}^7, \varphi_0)$ as vector spaces for every point $p \in M$. This is equivalent to a reduction of the tangent frame bundle from $GL(7, \mathbb{R})$ to the Lie group G_2 .

Remark. We will make frequent use of the fact that $d^*\varphi = 0$ if and only if $d\star\varphi = 0$. Also, because of the inclusion G_2 in $SO(7)$, all manifolds with G_2 -structure are necessarily orientable; further, it can be shown that all manifolds with G_2 -structure are spin, and any 7-manifold with spin structure admits a G_2 -structure.

A natural geometric requirement is that φ be constant with respect to the Levi-Civita connection of the G_2 -metric g_φ defined by φ . In this case, the holonomy of (M, φ) is a subgroup of G_2 , and (M, φ) is called a G_2 -manifold. The condition that $\nabla\varphi = 0$ is equivalent to $d\varphi = 0$ and $d^*\varphi = 0$ where d^* is the adjoint operator to the exterior derivative with respect to the Hodge star associated to the G_2 -metric g_φ . Fernandez and Gray [20] show that G_2 -manifolds are just 1 of 16 types of G_2 -structures on manifolds. Two of these classes include natural weakenings of

the G_2 -manifold requirements to manifolds with closed G_2 -structures, $d\varphi = 0$, and manifolds with coclosed G_2 -structures, $d^*\varphi = 0$.

In [15], we defined a G_2 -morphism to be a diffeomorphism $\Upsilon : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ of manifolds with G_2 -structures such that $\Upsilon^*(\varphi_2) = \varphi_1$. Because d commutes with pullback maps, we get for free that $d\varphi_1 = 0$ if and only if $d\varphi_2 = 0$.

Definition 1.2. Let $\Psi : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ be a diffeomorphism such that $\Psi^*(\star_2\varphi_2) = \star_1\varphi_1$. Then Ψ is called a *co G_2 -morphism*. Again, since d commutes with pullback maps, we have $d\star_1\varphi_1 = 0$ if and only if $d\star_2\varphi_2 = 0$.

Let (M_1, φ_1) and (M_2, φ_2) be two 7-dimensional manifolds with G_2 -structures. Let $M_1 \times M_2$ be the standard Cartesian product of M_1 and M_2 with canonical projection maps $\pi_i : M_1 \times M_2 \rightarrow M_i$. Define a 4-form $\star\varphi = \pi_1^*(\star_1\varphi_1) + \pi_2^*(\star_2\varphi_2)$. If both $\star_1\varphi_1$ and $\star_2\varphi_2$ are closed, then this form is also closed; in fact, for any $a_1, a_2 \in \mathbb{R}$, $a_1\pi_1^*(\star_1\varphi_1) + a_2\pi_2^*(\star_2\varphi_2)$ defines a (closed) 4-form on $M_1 \times M_2$. Taking $a_1 = 1$ and $a_2 = -1$, we have the (closed) 4-form $\widetilde{\star\varphi} = \pi_1^*(\star_1\varphi_1) - \pi_2^*(\star_2\varphi_2)$.

Theorem 1.3. A diffeomorphism $\Psi : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ is a *co G_2 -morphism* if and only if $\widetilde{\star\varphi}|_{\Gamma_\Psi} \equiv 0$, where $\Gamma_\Psi := \{(p, \Psi(p)) \in M_1 \times M_2 : p \in M_1\}$.

Proof. The submanifold Γ_Ψ is the embedded image of M_1 in $M_1 \times M_2$ with embedding given by $\tilde{\Psi} : M_1 \rightarrow M_1 \times M_2$ $\tilde{\Psi}(p) = (p, \Psi(p))$. Then $\widetilde{\star\varphi}|_{\Gamma_\Psi} = 0$ if and only if

$$\begin{aligned} 0 &= \tilde{\Psi}^*(\widetilde{\star\varphi}) = \tilde{\Psi}^*\pi_1^*(\star_1\varphi_1) - \tilde{\Psi}^*\pi_2^*(\star_2\varphi_2) \\ &= (\pi_1 \circ \tilde{\Psi})^*(\star_1\varphi_1) - (\pi_2 \circ \tilde{\Psi})^*(\star_2\varphi_2) = (id_{M_1})^*(\star_1\varphi_1) - \Psi^*(\star_2\varphi_2) = \star_1\varphi_1 - \Psi^*(\star_2\varphi_2). \end{aligned}$$

□

2. CO G_2 VECTOR FIELDS, COROCHESTERIAN 2-FORMS AND COROCHESTERIAN VECTOR FIELDS

Let M be a 7-manifold with a G_2 -structure. Recall that there is an action of the Lie group G_2 on the algebra of differential forms on M from which we obtain decompositions of each space of k -forms on M into irreducible G_2 -representations. In particular, we can decompose the space of 3-forms into the direct sum of a one-dimensional representation, a seven-dimensional representation and a 27-dimensional representation, denoted from here on by Ω_1^3 , Ω_7^3 and Ω_{27}^3 respectively; it is well known that $\Omega_7^3 = \{X \lrcorner \star\varphi : X \in \mathcal{X}(M)\}$ where $\mathcal{X}(M)$ is the space of vector fields on M (see for example [20, 25, 26, 31]).

Definition 2.1. Let (M, φ) be a manifold with coclosed G_2 -structure.

- (1) We define a *coRochesterian 2-form* σ to be any 2-form on M such that $d\sigma \in \Omega_7^3(M)$. We denote the set of coRochesterian 2-forms on M by $\Omega_{coRoc}^2(M)$.
- (2) For a coRochesterian 2-form σ , the vector field X_σ satisfying $X_\sigma \lrcorner \star\varphi = d\sigma$ will be called a *coRochesterian vector field*, and the set of coRochesterian vector fields on M will be denoted by $\mathcal{X}_{coRoc}(M)$.
- (3) A vector field X is called a *co G_2 -vector field* if $\mathcal{L}_X(\star\varphi) = 0$. $\mathcal{X}_{coG_2}(M)$ will denote the set of all co G_2 -vector fields.

That \mathcal{X}_{coG_2} , Ω_{coRoc}^2 and \mathcal{X}_{coRoc} are vector spaces follows immediately from the linearity properties of d and the interior product. We have the standard fact that X is a co G_2 -vector field if and only if $d(X \lrcorner \star\varphi) = 0$ since $d\star\varphi = 0$ implies that

$\mathcal{L}_X(\star\varphi) = d(X \lrcorner \star\varphi) + X \lrcorner d\star\varphi = d(X \lrcorner \star\varphi)$. Here, as in the case of closed G_2 -structures, the map $\widetilde{\star\varphi} : \mathcal{X}(M) \rightarrow \Omega^3(M)$ given by $\widetilde{\star\varphi}(X) = X \lrcorner \star\varphi$ cannot be an isomorphism; however, by the nondegeneracy condition on the 4-form $\star\varphi$, we do have that $\widetilde{\star\varphi}$ is injective, so for a given coRochesterian 2-form σ , the associated coRochesterian vector field X_σ is unique.

Theorem 2.2. *Every coRochesterian vector field on a manifold M with coclosed G_2 -structure φ is a $\text{co}G_2$ -vector field. If every closed form in $\Omega_7^3(M)$ is exact, then the spaces $\mathcal{X}_{\text{coRoc}}(M)$ and $\mathcal{X}_{\text{co}G_2}(M)$ coincide.*

Proof. The first statement follows immediately from the definitions. Next, for a $\text{co}G_2$ -vector field X , $X \lrcorner \star\varphi \in \Omega_7^3(M)$ is closed, so, by assumption, there exists a 2-form σ with $X \lrcorner \star\varphi = d\sigma$. \square

Corollary 2.3. *If $H^3(M) = \{0\}$, then every $\text{co}G_2$ -vector field on a manifold with coclosed G_2 -structure is a coRochesterian vector field.*

Proposition 2.4. *For any $\text{co}G_2$ -vector fields X_1, X_2 , there exists a 2-form σ such that $[X_1, X_2] \lrcorner \star\varphi = d\sigma$.*

Proof.

$$\begin{aligned} [X_1, X_2] \lrcorner \star\varphi &= \mathcal{L}_{X_1}(X_2 \lrcorner \star\varphi) - X_2 \lrcorner (\underbrace{\mathcal{L}_{X_1}(\star\varphi)}_{=0}) \\ &= \mathcal{L}_{X_1}(X_2 \lrcorner \star\varphi) = d(X_1 \lrcorner X_2 \lrcorner \star\varphi) + X_1 \lrcorner (\underbrace{d(X_2 \lrcorner \star\varphi)}_{=0}) \\ &= d(\star\varphi(X_2, X_1, \cdot)) \end{aligned}$$

Thus, $[X_1, X_2]$ is a coRochesterian vector field with an associated 2-form given by $\star\varphi(X_2, X_1, \cdot, \cdot)$. \square

Thus, we have the following inclusions of *Lie algebras*:

$$(\mathcal{X}_{\text{coRoc}}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}_{\text{co}G_2}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}(M), [\cdot, \cdot]).$$

For a coRochesterian 2-form σ , the assignment $\sigma \mapsto X_\sigma$ where X_σ is the unique associated coRochesterian vector field is linear. We now equip $\Omega_{\text{coRoc}}^2(M)$ with a bracket as follows: for $\sigma, \tau \in \Omega_{\text{coRoc}}^2(M)$, define $\{\sigma, \tau\} = \star\varphi(X_\sigma, X_\tau, \cdot, \cdot)$. Then $\{\sigma, \tau\} \in \Omega_{\text{coRoc}}^2(M)$ with coRochesterian vector field given by $[X_\tau, X_\sigma]$ since

$$d(\{\sigma, \tau\}) = d(\star\varphi(X_\sigma, X_\tau, \cdot, \cdot)) = [X_\tau, X_\sigma] \lrcorner \star\varphi.$$

Remark. This bracket is again a specific case of the *semibracket* defined in [4] for the general multisymplectic setting, and a proof of the following result in this more general setting can be found in [4, Proposition 3.7].

Proposition 2.5. *For any $\sigma, \tau, v \in \Omega_{\text{coRoc}}^2(M)$,*

$$\{\sigma, \{\tau, v\}\} + \{\tau, \{v, \sigma\}\} + \{v, \{\sigma, \tau\}\} = d(X_\sigma \lrcorner X_\tau \lrcorner dv)$$

Proof. Let $\sigma, \tau, v \in \Omega_{coRoc}^2(M)$ with associated Rochesterian vector fields X_σ, X_τ and X_v respectively. Then we have the following:

$$\begin{aligned}
& \{\sigma, \{\tau, v\}\} + \{\tau, \{v, \sigma\}\} + \{v, \{\sigma, \tau\}\} = \{\sigma, \{\tau, v\}\} - \{\tau, \{\sigma, v\}\} - \{\{\sigma, \tau\}, v\} \\
& = X_\sigma \lrcorner X_{\{\tau, v\}} \star \varphi - X_\tau \lrcorner X_{\{\sigma, v\}} \star \varphi - X_{\{\sigma, \tau\}} \lrcorner X_v \star \varphi \\
& = X_\sigma \lrcorner d\{\tau, v\} - X_\tau \lrcorner d\{\sigma, v\} + [X_\sigma, X_\tau] \lrcorner dv \\
& = X_\sigma \lrcorner d(X_v \lrcorner X_\tau \star \varphi) - X_\tau \lrcorner d(X_v \lrcorner X_\sigma \star \varphi) + [X_\sigma, X_\tau] \lrcorner dv \\
& = -X_\sigma \lrcorner d(X_\tau \lrcorner X_v \star \varphi) + X_\tau \lrcorner d(X_\sigma \lrcorner X_v \star \varphi) + [X_\sigma, X_\tau] \lrcorner dv \\
& = -X_\sigma \lrcorner d(X_\tau \lrcorner dv) + X_\tau \lrcorner d(X_\sigma \lrcorner dv) + [X_\sigma, X_\tau] \lrcorner dv \\
& = -X_\sigma \lrcorner d(X_\tau \lrcorner dv) + X_\tau \lrcorner d(X_\sigma \lrcorner dv) + \mathcal{L}_{X_\sigma}(X_\tau \lrcorner dv) - X_\tau \lrcorner (\mathcal{L}_{X_\sigma} dv) \\
& = -X_\sigma \lrcorner d(X_\tau \lrcorner dv) + X_\tau \lrcorner d(X_\sigma \lrcorner dv) + X_\sigma \lrcorner d(X_\tau \lrcorner dv) \\
& \quad + d(X_\sigma \lrcorner X_\tau \lrcorner dv) \underbrace{- X_\tau \lrcorner (X_\sigma \lrcorner ddv)}_{=0} - X_\tau \lrcorner d(X_\sigma \lrcorner dv) \\
& = X_\sigma \lrcorner d(X_\tau \lrcorner dv) - X_\sigma \lrcorner d(X_\tau \lrcorner dv) + X_\tau \lrcorner d(X_\sigma \lrcorner dv) - X_\tau \lrcorner d(X_\sigma \lrcorner dv) + d(X_\sigma \lrcorner X_\tau \lrcorner dv) \\
& = d(X_\sigma \lrcorner X_\tau \lrcorner dv)
\end{aligned}$$

□

While we do not have a Lie algebra structure on $\Omega_{coRoc}^2(M)$, we do, as noted above, still have a linear transformation $\Phi : \Omega_{coRoc}^2(M) \rightarrow \mathcal{X}_{coRoc}(M)$. Assume that $\Phi(\sigma) = X_\sigma = 0$, then $0 = X_\sigma \lrcorner \star \varphi = d\sigma$ which implies that σ is a closed 2-form. Hence, coRochesterian vector fields are uniquely defined by their coRochesterian 2-forms, up to the addition of a closed 2-form.

Theorem 2.6. (1) *Given two coRochesterian 2-forms $\sigma_1, \sigma_2 \in \Omega_{coRoc}^2(M)$, $\{\sigma_1, \sigma_2\} \in \ker \Phi$ if and only if $d\sigma_1$ is constant along the flow lines of X_{σ_2} if and only if $d\sigma_2$ is constant along the flow lines of X_{σ_1} .*

(2) *Let $\psi : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ be a diffeomorphism. Then ψ is a coG_2 -morphism if and only if $\psi^*(\{\sigma, \tau\}) = \{\psi^*\sigma, \psi^*\tau\}$ for all $\sigma, \tau \in \Omega_{coRoc}^2(M_2)$.*

Proof. (1) We show only the first equivalence since the second equivalence follows similarly. From the definition of the bracket, we have

$$\begin{aligned}
\{\sigma_1, \sigma_2\} &= \star \varphi(X_{\sigma_1}, X_{\sigma_2}, \cdot, \cdot) = X_{\sigma_2} \lrcorner (X_{\sigma_1} \lrcorner \star \varphi) \\
&= X_{\sigma_2} \lrcorner d\sigma_1 = \mathcal{L}_{X_{\sigma_2}} \sigma_1 - d(X_{\sigma_2} \lrcorner \sigma_1).
\end{aligned}$$

From this, we see that $d\{\sigma_1, \sigma_2\} = d\mathcal{L}_{X_{\sigma_2}} \sigma_1 = \mathcal{L}_{X_{\sigma_2}}(d\sigma_1)$. Then $\{\sigma_1, \sigma_2\} \in \ker \Phi$ if and only if $\mathcal{L}_{X_{\sigma_2}}(d\sigma_1) = 0$.

(2) First, assume that ψ is a coG_2 -morphism, and note that for $p \in M_1$, we have the maps

$$\begin{aligned}
d\psi_p &: T_p M_1 \rightarrow T_{\psi(p)} M_2 \\
\psi_p^* &: T_{\psi(p)}^* M_2 \rightarrow T_p^* M_1 \\
d\psi_{\psi(p)}^{-1} &= (d\psi_p)^{-1} : T_{\psi(p)} M_2 \rightarrow T_p M_1
\end{aligned}$$

Since ψ is a coG_2 -morphism, $\psi^*(\star_2 \varphi_2) = \star_1 \varphi_1$ and $(\psi^{-1})^*(\star_1 \varphi_1) = \star_2 \varphi_2$, so by definition, for $p \in M_1$, we then get the following equivalent equations

$$\begin{aligned}
(\star_1 \varphi_1)_p(\cdot, \cdot, \cdot, \cdot) &= \psi_p^*((\star_2 \varphi_2)_{\psi(p)})(\cdot, \cdot, \cdot, \cdot) = (\star_2 \varphi_2)_{\psi(p)}(d\psi_p \cdot, d\psi_p \cdot, d\psi_p \cdot, d\psi_p \cdot) \\
(\star_2 \varphi_2)_{\psi(p)}(\cdot, \cdot, \cdot, \cdot) &= (\psi_{\psi(p)}^{-1})^*((\star_1 \varphi_1)_p)(\cdot, \cdot, \cdot, \cdot) = (\star_1 \varphi_1)_p(d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot)
\end{aligned}$$

Thus, we calculate for a coRochesterian 2-form $\sigma \in \Omega_{coRoc}^2(M_2)$ and vector fields Y, Z, W on M_1 ,

$$\begin{aligned}
(X_{\psi^*\sigma} \lrcorner \star_1 \varphi_1)_p(Y_p, Z_p, W_p) &= d(\psi^*\sigma)_p(Y_p, Z_p, W_p) \\
&= \psi_p^*(d\sigma_{\psi(p)})(Y_p, Z_p, W_p) \\
&= \psi_p^*((X_{\sigma} \lrcorner \star_2 \varphi_2)_{\psi(p)})(Y_p, Z_p, W_p) \\
&= \psi_p^*((\star_2 \varphi_2)_{\psi(p)}((X_{\sigma})_{\psi(p)}, \cdot, \cdot, \cdot))(Y_p, Z_p, W_p) \\
&= (\star_2 \varphi_2)_{\psi(p)}((X_{\sigma})_{\psi(p)}, d\psi_p Y_p, d\psi_p Z_p, d\psi_p W_p) \\
&= (\star_1 \varphi_1)_p(d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}, d\psi_{\psi(p)}^{-1}(d\psi_p Y_p), d\psi_{\psi(p)}^{-1}(d\psi_p Z_p), d\psi_{\psi(p)}^{-1}(d\psi_p W_p)) \\
&= (\star_1 \varphi_1)_p(d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}, Y_p, Z_p, W_p)
\end{aligned}$$

that is, $(X_{\psi^*\sigma})_p = d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}$. Hence we find that

$$\begin{aligned}
(\psi^*\{\sigma, \tau\})_p(Y_p, Z_p) &= (\psi^*(\star_2 \varphi_2(X_{\sigma}, X_{\tau}, \cdot)))_p(Y_p, Z_p) \\
&= \psi_p^*((\star_2 \varphi_2)_{\psi(p)}((X_{\sigma})_{\psi(p)}, (X_{\tau})_{\psi(p)}, \cdot))(Y_p, Z_p) \\
&= (\star_2 \varphi_2)_{\psi(p)}((X_{\sigma})_{\psi(p)}, (X_{\tau})_{\psi(p)}, d\psi_p Y_p, d\psi_p Z_p) \\
&= (\star_1 \varphi_1)_p(d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}, d\psi_{\psi(p)}^{-1}(X_{\tau})_{\psi(p)}, d\psi_{\psi(p)}^{-1}(d\psi_p Y_p), d\psi_{\psi(p)}^{-1}(d\psi_p Z_p)) \\
&= (\star_1 \varphi_1)_p((X_{\psi^*\sigma})_p, (X_{\psi^*\tau})_p, Y_p, Z_p) \\
&= (\star_1 \varphi_1)_p((X_{\psi^*\sigma})_p, (X_{\psi^*\tau})_p, \cdot, \cdot)(Y_p, Z_p) = \{\psi^*\sigma, \psi^*\tau\}_p(Y_p, Z_p)
\end{aligned}$$

Conversely, assume that $\psi^*(\{\sigma, \tau\}) = \{\psi^*\sigma, \psi^*\tau\}$ for all $\sigma, \tau \in \Omega_{coRoc}^2(M_2)$. Then, for any $\sigma, \tau \in \Omega_{coRoc}^2(M_2)$ we have

$$\begin{aligned}
\psi^*(\{\sigma, \tau\}) &= \psi^*(\star_2 \varphi_2(X_{\sigma}, X_{\tau}, \cdot, \cdot)) = \psi^*(X_{\tau} \lrcorner X_{\sigma} \lrcorner \star_2 \varphi_2) = \psi^*(X_{\tau} \lrcorner d\sigma) \\
&= \psi^*(d\sigma(X_{\tau}, \cdot, \cdot)) = d\sigma(X_{\tau}, d\psi \cdot, d\psi \cdot) = d\sigma(d\psi(d\psi^{-1} X_{\tau}), d\psi \cdot, d\psi \cdot) \\
&= (\psi^* d\sigma)(d\psi^{-1} X_{\tau}, \cdot, \cdot) = (d\psi^{-1} X_{\tau}) \lrcorner (\psi^* d\sigma)
\end{aligned}$$

and

$$\{\psi^*\sigma, \psi^*\tau\} = \star_1 \varphi_1(X_{\psi^*\sigma}, X_{\psi^*\tau}, \cdot, \cdot) = X_{\psi^*\tau} \lrcorner d(\psi^*\sigma) = X_{\psi^*\tau} \lrcorner (\psi^* d\sigma)$$

which, by our hypothesis, yields that $d\psi^{-1} X_{\tau} = X_{\psi^*\tau}$ for any $\tau \in \Omega_{coRoc}^2(M_2)$. Then for any $\sigma \in \Omega_{coRoc}^2(M_2)$, any vector fields $Y, Z, W \in \mathcal{X}(M_1)$ and $p \in M_1$,

$$\begin{aligned}
(X_{\psi^*\sigma} \lrcorner \star_1 \varphi_1)_p(Y_p, Z_p, W_p) &= d(\psi^*\sigma)_p(Y_p, Z_p, W_p) = (\psi^* d\sigma)_p(Y_p, Z_p, W_p) \\
&= \psi_p^*(X_{\sigma} \lrcorner \star_2 \varphi_2)_p(Y_p, Z_p, W_p) \\
&= (\star_2 \varphi_2)_{\psi(p)}((X_{\sigma})_{\psi(p)}, d\psi_p Y_p, d\psi_p Z_p, d\psi_p W_p) \\
&= (\star_2 \varphi_2)_{\psi(p)}(d\psi_p(d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}), d\psi_p Y_p, d\psi_p Z_p, d\psi_p W_p) \\
&= (\psi^*(\star_2 \varphi_2))_{\psi(p)}(d\psi_{\psi(p)}^{-1}(X_{\sigma})_{\psi(p)}, Y_p, Z_p, W_p) \\
&= (\psi^*(\star_2 \varphi_2))_{\psi(p)}((X_{\psi^*\sigma})_p, Y_p, Z_p, W_p)
\end{aligned}$$

Thus $X_{\psi^*\sigma} \lrcorner \star_1 \varphi_1 = X_{\psi^*\sigma} \lrcorner \psi^*(\star_2 \varphi_2)$ which implies that $\star_1 \varphi_1 = \psi^*(\star_2 \varphi_2)$ as desired. \square

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